the principal term of the asymptotic solution is continuous for all $t \geqslant 0, x \geqslant 0$.

The technique can also be used formally in the case when the functions f_i in (4.1), g_i in (4.2) and h_j in (4.3) depend on the slow variables τ, ξ, or non-linear conditions that are solved with respect to the functions $u_j(t, 0), j = 1, \ldots, m$ are considered instead of the linear boundary conditions (4.3).

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EQUATIONS DESCRIBING THE PROPAGATION OF NON-LINEAR QUASITRANSVERSE WAVES IN A WEAKLY NON-ISOTROPIC ELASTIC BODY*

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Approximate equations are obtained, describing the propagation of a non-linear quasitransverse wave of low amplitude, or a group of such waves, in a nearly isotropic elastic medium, when the characteristic velocities of the waves (dependent on their polarization) differ from one another by a small quantity.

The equations of non-linear geometrical acoustics, and the short-wave equations, are well-known /1-9/; they are obtained on the basis of the fact that waves connected with one family of characteristic surfaces can be propagated. Disturbances linked with other characteristics interact weakly with these waves, by virtue of the assumptions that the amplitude is small and the waves are quasiplane. It is also important that, due to the small difference in the wave velocities, their interaction time is small, if the length of the groups of waves in question is finite.

With small anisotropy and non-linearity, the equations of the theory of elasticity have two properties: the two characteristic velocities corresponding to quasitransverse waves are close, and the non-linearity is extremely small. In the absence of anisotropy (including that due to initial deformation), the non-linearity appears only in the cubic terms; while if there is small anisotropy, quadratic terms also make an appearance, though with small coefficients. Due to the closeness of the quasi-transverse wave velocities, they interact together long-term, so that the evolution of these waves can be studied by considering two waves simultaneously.

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The remaining waves (given suitable assumptions about their initial amplitudes being small) can be excluded from consideration. The small non-linearity of the equations leads to the need to take a large number of terms into account for a correct description of the effects of non-linearity.

The approximate equation for non-linear plane-polarized quasi-transverse elastic waves (connected with one family of characteristics of quasitransverse waves), propagating in an undeformed isotropic elastic medium, was obtained in /10/. The paper by Cohen and Kulsrud /11/ is closest in its subject to the present paper; in the context of magneto-hydrodynamics, they obtained an approximate system of equations, describing the purely one-dimensional quasitransverse waves with any polarization, propagating along the magnetic field.

1. One-dimensional waves on a homogeneous background. The equations of one-dimensional motions in the theory of elasticity may be written as

$$\rho_0 \frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial u_i}\right) \equiv f_{ij}\frac{\partial u_j}{\partial x} \tag{1.1}$$

$$\frac{\partial u_i}{\partial t} = \frac{\partial v_i}{\partial x}, \quad f_{ij} = \frac{\partial^2 \Phi}{\partial u_i \partial u_j} \equiv f_{ij}^\circ + g_{ij}$$

$$g_{ij} = \frac{\partial^2 P}{\partial u_i \partial u_j}, \quad P = \Phi - \frac{1}{2}\mu(u_1^2 + u_2^2) - \frac{1}{2}(\lambda+\mu)u_3^2$$

Here, $v_i = \partial w_i/\partial t$, $u_i = \partial w_i/\partial x$, w_i are the displacements of particles, considered as functions of Lagrangian Cartesian coordinates $x_1, x_2, x_3 = x$, ρ_0 is the initial density in these coordinates, f_{ij}° is the matrix corresponding to the linear isotropic medium: $f_{11}^\circ = \mu$, $f_{22}^\circ = \mu$, $f_{33}^\circ = \lambda + 2\mu$, $f_{ij}^\circ = 0$ for $i \neq j$. We assume that Φ can be expanded in power series in u_j, where the expansion coefficients differ from the corresponding expansion for an isotropic body by quantities of the order of δ ($\delta \ll 1$ is the anisotropy parameter).

We shall consider quasitransverse waves, in which $u_3 - u_3^\circ \lesssim \varepsilon^2$, $\varepsilon = \max\{(u_1 - u_1^\circ), (u_2 - u_2^\circ)\}$, where the superscript $^\circ$ refers to the state before the wave. It follows from the form of Φ that $g_{33} \sim g_{\alpha\beta} \lesssim \chi$ ($\alpha, \beta = 1, 2$) $\chi = \max\{\varepsilon^2, \delta\}$, $g_{\alpha 3} \leqslant \max\{\varepsilon, \delta\}$, and the characteristic velocity of the quasitransverse waves is equal to $\sqrt{\mu/\rho_0} + O(\chi)$. Hence, with a relative error $O(\chi)$, all the quantities in the quasitransverse wave, and in particular u_3, satisfy the relation

$$L_0 u_3 = 0, \quad L_0 \equiv \frac{\partial}{\partial t} + \sqrt{\frac{\mu}{\rho_0}}\frac{\partial}{\partial x} \tag{1.2}$$

From (1.1) with $i = 3$ we obtain

$$v_3 = -\sqrt{\mu/\rho_0}\, u_3 + \text{const} \tag{1.3}$$

$$\frac{\partial u_3}{\partial x} = -\frac{1}{\lambda+\mu}\left(g_{13}\frac{\partial u_1}{\partial x} + g_{23}\frac{\partial u_2}{\partial x}\right)$$

$$u_3 = -\frac{1}{\lambda+\mu}\frac{\partial P}{\partial u_3} + u_3^\circ, \quad u_3^\circ = \text{const}$$

These and all subsequent equations have a realtive error with respect to the terms written which does not exceed χ. Obviously, the above operation of finding u_3 consists in finding the forced solution of Eqs.(1.1) for u_3, linked with the transverse waves. The free solution, corresponding to longitudinal waves, is ignored here; this is permissible when the initial conditions, corresponding to these waves, are zero.

Using the equation obtained in the first two of Eqs.(1.1), we obtain

$$\rho_0 \frac{\partial v_\alpha}{\partial t} = (\mu\delta_{\alpha\beta} + h_{\alpha\beta})\frac{\partial u_\beta}{\partial x} \equiv \frac{\partial}{\partial x}\frac{\partial Q}{\partial u_\alpha} \tag{1.4}$$

$$h_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{\lambda+\mu}\overset{\circ}{g}_{\alpha 3}\overset{\circ}{g}_{\beta 3} \equiv \frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}$$

$$Q = \frac{1}{2}\mu(u_1^2 + u_2^2) + F(u_1, u_2)$$

$$F = P^\circ - \frac{1}{2(\lambda+\mu)}\left(\frac{\partial P}{\partial u_3}\right)^{\circ 2}$$

Here and below, the Greek subscripts take the values 1, 2, and the Latin subscripts the values 1, 2, 3. The superscript $^\circ$ means that the relevant quantity is taken with $u_3 = u_3^\circ$. The equations obtained serve to describe the quasitransverse waves. The longitudinal gradient of the displacement u_3 can be found from Eq.(1.3).

To obtain the equations describing the propagation of transverse waves only in the position direction of the x axis, we make a change of variables, and introduce as required functions the invariants which are preserved in linear waves in an isotropic medium, and travel to the right (superscript plus) and to the left (superscript minus)

$$u_\alpha^+ = {}^1/_2\left(u_\alpha - \sqrt{\frac{\mu}{\rho_0}}\, v_\alpha\right), \quad u_\alpha^- = {}^1/_2\left(u_\alpha + \sqrt{\frac{\mu}{\rho_0}}\, v_\alpha\right) \tag{1.5}$$

On multiplying Eqs.(1.4) by $1/\sqrt{\mu\rho_0}$ and taking the half-difference with the fourth and fifth equations of (1.1), we obtain

$$\frac{\partial u_\alpha^+}{\partial t} + \sqrt{\frac{\mu}{\rho_0}}\frac{\partial u_\alpha^+}{\partial x} + \frac{h_{\alpha\beta}}{2\sqrt{\mu\rho_0}}\frac{\partial}{\partial x}(u_\beta^+ + u_\beta^-) = 0 \tag{1.6}$$

In the same way, taking the half-sums of the same equations, we obtain the equations for u_α^-. We find from these last equations that the forced solution u_α^-, linked with the wave travelling to the right (i.e., satisfying a relation of the type (1.2) $L_0 u_\alpha^- = 0$), is of the order of hu_γ, where $h = \max\{h_{\alpha,\beta}\} \sim \chi$. Hence, by the relations $u_\alpha = u_\alpha^+ + u_\alpha^-$, $L_0 u_\alpha^- = 0$, the superscript plus in the first two terms of Eq.(1.6) can be dropped with tolerable accuracy, i.e.,

$$\frac{\partial u_\alpha}{\partial t} + \sqrt{\frac{\mu}{\rho_0}}\frac{\partial u_\alpha}{\partial x} + \frac{h_{\alpha\beta}}{2\sqrt{\mu\rho_0}}\frac{\partial u_\beta}{\partial x} = 0 \tag{1.7}$$

The equations obtained, along with (1.3) for u_3, describe quasitransverse elastic waves of low amplitude, travelling in the positive direction of the x axis. Notice that both (1.4) and (1.7) have the divergence form, corresponding to which we have the integral form of the relations at a discontinuity

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial u_\alpha}\right) = 0 \tag{1.8}$$

$$R = \frac{1}{2}\sqrt{\frac{\mu}{\rho_0}}(u_1^2 + u_2^2) + \frac{1}{2\sqrt{\mu\rho_0}} F(u_1, u_2)$$

$$\left(\sqrt{\frac{\mu}{\rho_0}} - W\right)[u_\alpha] + \frac{1}{2\sqrt{\mu\rho_0}}\left[\frac{\partial F}{\partial u_\alpha}\right] = 0 \tag{1.9}$$

(F is given by Eqs.(1.4), and the brackets denote jumps in the quantities in the brackets). The last relation is the same, up to the given accuracy, as the condition for conservation of transverse momentum, which can be obtained as the integral form of Eq.(1.4) (these conditions are the same if, when finding the velocity of discontinuity W, no account is taken of a quantity of the second order of smallness $(W - \sqrt{\mu/\rho_0})^2$).

Relation (1.9) can also be jusitifed as follows. We assume that there are extra terms $\partial\tau_{i3}/\partial x$ on the right-hand sides of the equations of motion, where τ_{i3} are components of the viscous stress tensor. We assume that these viscous terms are small compared with the principal terms of Eqs.(1.1) (in many problems of mechanics, notably in the problem of the stucture of a shock wave, these terms are of the same order as the non-linear terms) and that they vanish along with the deformation rate tensor. Then, on again performing all the calculations, we find that there appear in Eqs.(1.8) the viscous terms

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial u_\alpha} + \frac{1}{2\sqrt{\mu\rho_0}}\tau_{\alpha 3}\right) = 0 \tag{1.10}$$

If a solution of the shock-wave structure type is considered, then (1.10) can again be integrated with respected to x from $-\infty$ to ∞. Since the $\tau_{\alpha 3}$ vanish at the ends of the integration along with the deformation rates, we obtain as a result relations (1.9).

Notice incidentally that, if the initial system of equations (1.1) is augmented by any terms f_i, small compared with the principal terms, on the right-hand sides, then the terms $(2\sqrt{\mu\rho_0})^{-1} f_i$ appear on the left-hand sides of Eqs.(1.10).

Being a hyperbolic system of two equations, (1.7) can be written in terms of Riemann invariants /12/

$$\frac{\partial I_\alpha}{\partial t} + c_\alpha(I_1, I_2)\frac{\partial I_\alpha}{\partial x} = 0 \tag{1.11}$$

Here, I_α is the first integral of the equation

$$\frac{du_2}{du_1} = \frac{b_2^\alpha(u_1, u_2)}{b_1^\alpha(u_1, u_2)},$$

while b_1^α, b_2^α are the components of the eigenvector of the matrix $\sqrt{\mu/\rho_0}\delta_{\beta\gamma} + h_{\alpha\beta}/2\sqrt{\mu\rho_0}$, corresponding to the eigenvalue c_α, representing the characteristic velocity of system (1.7). In Eqs.(1.11), this characteristic velocity is regarded as a function of I_1 and I_2.

2. Weakly non-one-dimensional waves in a weakly non-homogeneous medium.

We shall study waves similar to those considered in Sect.1, but dependent only on $x_3 = x$. As in Sect.1, we put $u_k = \partial w_k/\partial x$. We shall assume that differentiation with respect to the transverse coordinates x_1, x_2 increases the order of smallness of the terms by a small factor η. The difference in the velocities of small disturbances, connected with the anisotropy of the medium, will be characterized by a smallness parameter δ. We will use the small factor ϑ to characterize the inhomogeneity of the properties of the medium with respect to the coordinates. The quantities $1/\eta$ and $1/\vartheta$ are characteristic lengths. We assume that $1/\eta \lesssim 1/\vartheta$. The characteristic scale of the unit wave and of the entire group of waves in their direction of propagation is the same (of the order of unity).

We shall preserve in the equations the terms of order ε, $\vartheta\varepsilon$, $\delta\varepsilon$, $\eta\varepsilon$, $\eta^2\varepsilon$, $\eta\varepsilon^2$, ε^3. The orders of the retained terms is chosen fairly arbitrarily, and can easily be varied so as to take account of higher terms. The orders here are such that all the terms used in /13, 14/ are retained. By taking account of terms of the order of ε^3, a correct description of non-linear processes can be ensured. Linear terms of the order of $\eta^2\varepsilon$, containing second derivatives with respect to the transverse coordinates, are needed e.g., to describe the damping of the waves as they diverge. The terms of the order of $\eta\varepsilon^2$ characterize the variation of the non-linear effects for waves, the normals to which are directed almost along the x axis. The terms of the order of $\vartheta\varepsilon$ in the crudest approximation take account of the effect of the inhomogeneity of the properties of the medium.

When estimating the orders, we shall assume that the initial gradients of the displacements $\partial w_\beta^\circ/\partial x_\alpha$ $(\alpha, \beta = 1,2)$ have a higher order than ε. This assumption is not essential, but on the one hand it simplifies later estimates, and on the other, it implies no constraints in general on the statement of the problem. This connected with the fact that, as was shown in /13, 14/, the difference between the characteristic velocities, the order of which was denoted above by δ, is order-wise equal to $\max\{\partial w_\beta^\circ/\partial x_\alpha\}$ in an isotropic medium. In general, therefore, we can assume that δ is greater than, or of the same order as, $\partial w_\beta^\circ/\partial x_\alpha$. Since, according to the stipulation made above, the terms with δ will only be taken into account in combinations $\varepsilon\delta$, this in fact implies that $\partial w_\beta^\circ/\partial x_\alpha$ is only taken into account in the lowest approximation, as if this quantity were of the order of ε^2 or $\eta\varepsilon$. To be specific, we shall henceforth assume the latter estimate.

We write the equations of the theory of elasticity in the three-dimensional case

$$\rho_0 \frac{\partial^2 w_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial w_{i,j}}\right) \equiv \frac{\partial^2 \Phi}{\partial w_{i,j}\partial w_{m,n}} w_{m,nj} + \frac{\partial'}{\partial x}\frac{\partial \Phi}{\partial w_{i,j}} \tag{2.1}$$

$$w_{i,j} = \frac{\partial w_i}{\partial x_j}, \quad w_{m,nj} = \frac{\partial^2 w_m}{\partial x_n \partial x_j}$$

Here, w_i are the particle displacements with respect to fixed Cartesian coordinates x_1, x_2, $x_3 = x$. The values of these coordinates, corresponding to the unstressed state, will be taken to be Lagrangian. The function Φ depends on $w_{m,n}$ and x. The symbol $\partial'/\partial x_j$ denotes differentiation with respect to x_j with constant $w_{m,n}$.

We shall write (2.1), corresponding to $i = 3$, in the crudest approximation, i.e., in the same way as in the case of a homogeneous isotropic medium; on additionally neglecting terms of relative order η^2 and replacing $\partial^2/\partial t^2$ by $\mu\rho_0^{-1}\partial^2/\partial x^2$ (the last operation can lead to a relative error ε^2 or $\eta\varepsilon$), we obtain

$$\frac{\partial^2 w_3}{\partial x^2} = -\frac{\partial^2 w_\alpha}{\partial x_\alpha \partial x} - \frac{g_{\alpha 3}}{\lambda+\mu}\frac{\partial^2 w_\alpha}{\partial x^2}\,.$$

The last term in this equation, as in (1.3), can be written as the derivative of a function with respect to x. Since only principal terms of order ε need be taken into account in $g_{\alpha 3}$ (if we took account of terms of, say, order δ in $g_{\alpha 3}$, we should arrive in the final equations at terms of order $\delta\varepsilon^2$, which we neglect in our approximation), these expressions thus have the same form as in weakly non-linear one-dimensional waves in an isotropic medium /13, 14/, $g_{\alpha 3} = 2b\partial w_\alpha/\partial x$. The constant b is expressible in terms of the coefficients of the expansion of Φ with respect to invariants of the deformation tensor and is given below (Eq. (2.6)).

Using the above equation for $g_{\alpha 3}$, the relation for $\partial^2 w_3/\partial x^2$ can be integrated with respect to x:

$$\frac{\partial w_3}{\partial x} = -\frac{\partial w_\alpha}{\partial x_\alpha} - \frac{b}{\lambda+\mu}\left[\left(\frac{\partial w_1}{\partial x}\right)^2 + \left(\frac{\partial w_2}{\partial x}\right)^2\right] + \mathrm{const} \tag{2.2}$$

By using Eq.(2.2), in the same way as Eq.(1.3) in the one-dimensional case, we can eliminate $\partial w_3/\partial x$ from the equations. In addition, the equation shows that the variation in the wave of the first invariant of the deformation tensor I_1 is of the order of ε^2, while the variation of w_3 is of the order of $\max\{\eta\varepsilon, \varepsilon^2\}$.

Now take Eqs.(2.1) for w_α $(\alpha = 1,2)$. The terms of orders ε^3, ε^2, ε on the right-hand sides are the same as the corresponding terms in Eqs.(1.1), describing the one-dimensional motions. The terms of orders $\eta\varepsilon$ and $\eta^2\varepsilon$ are contained in the linear equations for the isotropic body and have the form

$$(\lambda+\mu)\frac{\partial w_{i,i}}{\partial x_\alpha}+\mu\Delta_\perp w_\alpha,\quad \Delta_\perp=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\,.$$

(We do not write here the term $\partial^2 w_\alpha/\partial x^2$, which does not contain η). If relation (2.2) is used, the above expression becomes

$$-b\frac{\partial}{\partial x_\alpha}\left[\left(\frac{\partial w_1}{\partial x}\right)^2+\left(\frac{\partial w_2}{\partial x}\right)^2\right]+\mu\Delta_\perp w_\alpha \tag{2.3}$$

The first term here is of the order of $\eta\varepsilon^2$, while there are no terms of the order of $\eta\varepsilon$.

The other terms of the order of $\eta\varepsilon^2$, contained in the first term on the right-hand side of (2.1), can be written as

$$\left(\frac{\partial^2\Phi}{\partial w_{\alpha,3}\partial w_{\beta,\gamma}}+\frac{\partial^2\Phi}{\partial w_{\beta,3}\partial w_{\alpha,\gamma}}\right)w_{\beta,\gamma 3} \tag{2.4}$$

Here we omit the terms which would be obtained if γ were equal to three, since they do not contain η (the corresponding to one-dimensional waves). In addition, we note that, by (2.2), the variation of w_3 in the wave is of the order of $\max\{\eta\varepsilon, \varepsilon^2\}$, and hence, to obtain the terms of the order of $\eta\varepsilon^2$, we have to retain in (2.1) only those terms in which there is not more than one index equal to three among the indices of $w_{m,nj}$.

Since $w_{\beta,\gamma 3}$ is of the order of $\eta\varepsilon$, it is sufficient to take account of the terms of the order of ε in the parentheses in (2.4). This means that Φ can be taken in the same form as in an isotropic medium and we can confine ourselves to terms up to the third degree in $w_{i,j}$

$$\Phi=\frac{1}{2}\lambda I_1^2+\mu I_2+\beta I_1 I_2+\gamma I_3,\quad I_1=\varepsilon_{ii} \tag{2.5}$$
$$I_2=\varepsilon_{ij}\varepsilon_{ij},\quad I_3=\varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki},\quad \varepsilon_{ij}=\frac{1}{2}(w_{i,j}+w_{j,i}+w_{k,i}\,w_{k,j})$$

Simple calculations lead to the following expressions for the terms of the order of $\eta\varepsilon^2$ in the equation for w_α:

$$\left(\lambda+\mu+\frac{3}{4}\gamma+2\beta\right)(w_{\alpha,3}w_{\beta,\beta 3}+w_{\beta,3}w_{\beta,\alpha 3})+$$
$$\left(2\mu+\frac{3}{2}\gamma\right)w_{\gamma,3}w_{\alpha,\gamma 3}\ .$$

Terms of the order of $\vartheta\varepsilon$ can only be contained in the linear part of the last term on the right-hand side of (2.1). There is one such term: $2w_{\alpha,3}\,\partial\mu/\partial x$.

Augmenting Eqs.(1.4) by terms of the orders of $\eta^2\varepsilon$, $\eta\varepsilon^2$, $\vartheta\varepsilon$, found above, we obtain the system of equtions for w_α:

$$\begin{aligned}\rho_0\frac{\partial^2 w_\alpha}{\partial t^2}&=(\mu\delta_{\alpha\beta}+h_{\alpha\beta})\frac{\partial^2 w_\beta}{\partial x^2}+2a\frac{\partial w_\alpha}{\partial x}\frac{\partial^2 w_\beta}{\partial x_\beta\partial x}+\\ &(a-b)\frac{\partial}{\partial x_\alpha}\left[\left(\frac{\partial w_1}{\partial x}\right)^2+\left(\frac{\partial w_2}{\partial x}\right)^2\right]+2c\frac{\partial w_\gamma}{\partial x}\frac{\partial^2 w_\alpha}{\partial x_\gamma\partial x}+\\ &\mu\left(\frac{\partial^2 w_\alpha}{\partial x_1^2}+\frac{\partial^2 w_\alpha}{\partial x_2^2}\right)+2\frac{\partial\mu}{\partial x}\frac{\partial w_\alpha}{\partial x}\\ 2a&=\gamma+\mu+{}^3/_4\gamma+2\beta,\quad 2b=\gamma+2\mu+\beta+{}^3/_2\gamma,\\ 2c&=2\mu+{}^3/_2\gamma\end{aligned} \tag{2.6}$$

For the case of an elastic medium with anisotropy, caused only be small (of the order of ε^2) initial deformations $\mathring{\varepsilon}_{\alpha\beta}$ (as in /13, 14/), $h_{\alpha\beta}$ have the form

$$h_{\alpha\beta}=\frac{\partial^2 F}{\partial u_\alpha\partial u_\beta}\,,\quad F=\frac{1}{2}(f_1u_1^2+f_2u_2^2)-\frac{\varkappa}{8}(u_1^2+u_2^2)^2$$
$$f_1=2b(\mathring{\varepsilon}_{11}+\mathring{\varepsilon}_{33})+(\lambda+\mu)\mathring{\varepsilon}_{22},\quad f_2=2b(\mathring{\varepsilon}_{22}+\mathring{\varepsilon}_{33})+(\lambda+\mu)\mathring{\varepsilon}_{11}$$
$$\varkappa=\mu+\frac{(\mu+\beta+{}^3/_2\gamma)^2}{\lambda+\mu}-2\xi,\quad u_\alpha=\frac{\partial w_\alpha}{\partial x}\ .$$

Here, ξ is the coefficients of I_2^2 in the expansion of Φ. We assume that $\varepsilon_{12}^\circ=0$ by virtue of the chosen directions of the x_1 and x_2 axes, while $\varepsilon_{\alpha 3}^\circ=0$ by virtue of the chosen origin of coordinates for u_1 and u_2.

Further, as in Sect.1, we obtain the equations describing the waves propagating only in the positive direction of the x axis. We introduce the velocity $v_\alpha=\partial w_\alpha/\partial t$. Obviously,

$$\frac{\partial v_\alpha}{\partial x}=\frac{\partial u_\alpha}{\partial t} \tag{2.7}$$

On introducing the variables u_α^+ and u_α^-, given by Eqs.(1.5), and taking in the same way as in Sect.1 linear combinations of Eqs.(2.6), the left-hand side of which is written like $\partial v_\alpha/\partial t$, and (2.7), we obtain for w_α, connected with the waves propagating in the positive direction of x axis

$$\frac{\partial^2 w_\alpha}{\partial t\partial x}+\sqrt{\frac{\mu}{\rho_0}}\frac{\partial^2 w_\alpha}{\partial x^2}+\frac{1}{\sqrt{\mu\rho_0}}\left[\frac{h_{\alpha\beta}}{2}\frac{\partial^2 w_\beta}{\partial x^2}+a\frac{\partial w_\alpha}{\partial x}\frac{\partial^2 w_\beta}{\partial x_\beta\partial x}+\right.$$
$$\left.(a-b)\frac{\partial w_\beta}{\partial x}\frac{\partial^2 w_\beta}{\partial x_\alpha\partial x}+c\frac{\partial w_\gamma}{\partial x}\frac{\partial^2 w_\alpha}{\partial x_\gamma\partial x}+\frac{\mu}{2}\left(\frac{\partial^2 w_\alpha}{\partial x_1^2}+\frac{\partial^2 w_\alpha}{\partial x_2^2}\right)+\frac{\partial\mu}{\partial x}\frac{\partial w_\alpha}{\partial x}\right]=0 \tag{2.8}$$

Notice that only the penultimate term contains w_α, not differentiated with respect to x. Under the same assumptions as in the ordinary case of non-linear geometrical acoustics /1-5/, we can express this term in terms of $\partial w_\alpha/\partial x$. We assume that, for our quasitransverse waves, a family of surfaces can be selected in such a way that w_1 and w_2 vary rapidly along the normal to the family (in the x direction) and vary very slowly on the surfaces themselves, which will henceforth be called wave surfaces. Then, neglecting the variations of w_α on the wave surfaces, we find that, in Cartesian coordinates x_1, x_2, x we have the equation

$$\frac{\partial^2 w_\alpha}{\partial x_1^2}+\frac{\partial^2 w_\alpha}{\partial x_1^2}=k\frac{\partial w_\alpha}{\partial x} \tag{2.9}$$

where $k=R_1^{-1}R_2^{-1}$ is the Gaussian curvature of the wave surface passing through the given point, and R_1, R_2 are the principal radii of curvature of the surface; k is assumed positive if the surface is seen as convex in the direction of wave propagation.

In order for the above equation to hold, the characteristic transverse dimension $l_\perp$ of the wave pencil must satisfy the condition $l_\perp^2\gg Rl$, where l is the wavelength, and R is the characteristic radius of curvature of the surface. This last approximation is typical for geometrical acoustics.

In many problems, e.g., when studying divergent waves, our wave surface can be regarded as moving, in the same way as when non-linearity and anisotropy are neglected, i.e., at velocities equal to $\sqrt{\mu/\rho_0}$. The position of these surfaces can then be calculated at all instants, and k becomes a known function of x in curvilinear coordinates with the x axis orthogonal to the surfaces. For a homogeneous medium, the x axis is a straight line and $k=k(x)=(R_{10}+x)^{-1}(R_{20}+x)^{-1}$, where R_{10}^{-1} and R_{20}^{-1} are the initial values of the principal curvatures. Then, system (2.8) takes the form

$$\frac{\partial u_\alpha}{\partial t}+\sqrt{\frac{\mu}{\rho_0}}\frac{\partial u_\alpha}{\partial x}+\frac{1}{\sqrt{\mu\rho_0}}\left[\frac{h_{\alpha\beta}}{2}\frac{\partial u_\beta}{\partial x}+au_\alpha\frac{\partial u_\beta}{\partial x_\beta}+(a-b)u_\beta\frac{\partial u_\beta}{\partial x_\alpha}+cu_\gamma\frac{\partial u_\alpha}{\partial x_\gamma}+ku_\alpha+\frac{\partial\mu}{\partial x}u_\alpha\right]=0 \tag{2.10}$$

The indices α, β, γ take the values 1 and 2. Summation is performed with respect to repeated indices. If $\sqrt{\mu/\rho_0}\neq\text{const}$, it is best to transform Eq.(2.8) or (2.10) from Cartesian to curvilinear coordinates, where the coordinate lines x_1 and x_2 lie on the wave surfaces, and the coordinate line x orthogonal to them. It is also possible to write the equations at reference points accompanying the waves, as was done in /8, 9/. If the wave surfaces can be found in advance, the equations are transformed in accordance with the rules of vector calculus.

Notice that, if the characteristic transverse dimension $l_\perp$ of the wave pencil is of the order of the radius of curvature R of the wave surfaces, then the terms containing differentiation with respect to x_1 and x_2 in (2.10) are of the order of ε with respect to the term that contains k, and they can be neglected, unless effects connected with these terms need to be specially investigated.

Our results can be extended to the case of a medium which is almost transversely isotropic, with anisotropy axis directed along the wave vector of the group of waves in question.

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SPATIAL INTERACTION OF STRONG DISCONTINUITIES IN A GAS*

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The spatial problem of the interaction of curved fronts of strong discontinuities during collision is examined for the system of gas-dynamic equations. In the case of regular interaction, an algorithm is indicated for the construction, and the existence of a piecewise-analytic solution of the problem in an exact formulation is proved. The series governing the solution converge in a certain neighbourhood of a two-dimensional surface γ_0 in the space $R^4(x, t)$, which is the intersection of surfaces of interacting discontinuities. It is shown that the solution cannot be piecewise-analytic in the neighbourhood of those points of γ_0 for which the normal velocity of the curve γ_{0t} with respect to the gas (a section through γ_0 by the plane $t = \text{const}$) is subsonic.

1. Formulation of the problem. For $t \in [-t_1, t_1]$ (t is the time), let an analytic solution $\mathbf{u} = \mathbf{u}_0(\mathbf{x}, t)$, $p = p_0(\mathbf{x}, t)$, $\rho = \rho_0(\mathbf{x}, t)$ of the system of gas-dynamics equations

$$\rho_t + \operatorname{div} \rho\mathbf{u} = 0, \quad (\rho u_l)_t + \operatorname{div}(\rho u_l \mathbf{u}) + (\nabla p)_l = 0 \tag{1.1}$$
$$(\rho(\varepsilon + {}^1/_2|\mathbf{u}|^2))_t + \operatorname{div} \rho\mathbf{u}(i + {}^1/_2|\mathbf{u}|^2) = 0 \quad (l = 1, 2, 3)$$

be known in the domain $\Omega \subset R^4(x, t)$ ($\mathbf{x} = (x_1, x_2, x_3) \in R^3$, $t \in R$) ($\mathbf{u} = (u_1, u_2, u_3)$ is the velocity vector, ρ is the density, p is the pressure, ε is the specific internal energy, and $i = \varepsilon + p\rho^{-1}$ is the specific enthalpy). The functions $\varepsilon = \varepsilon(v, p)$, $p = g(v, s)$ (here $v = \rho^{-1}$ and s is the entropy) that give the equation of state of the medium are analytic and satisfy the normal gas conditions /1/. The fronts of two strong discontinuities propagate over the background "null", where the surfaces of discontinuity $\Gamma_j \subset R^4(\mathbf{x}, t)$ and the solutions behind the fronts $\mathbf{u} = \mathbf{u}_j(\mathbf{x}, t)$, $p = p_j(\mathbf{x}, t)$, $\rho = \rho_j(\mathbf{x}, t)$ ($j = 1, 2$) are analytic. (The discontinuities are concentrated on the hypersurfaces Γ_j in the space $R^4(\mathbf{x}, t)$. Sections Γ_{jt} through these surfaces by the planes

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